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Non-trivial stably free modules over crossed products

Natalia Iyudu[1] **and Robert Wisbauer**[2]

[1] Department of Pure Mathematics, Queen's University Belfast, UK
[2] Mathematisches Institut, Heinrich Heine Universität, 40225 Düsseldorf, Germany

E-mail: n.iyudu@qub.ac.uk and wisbauer@math.uni-duesseldorf.de

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Abstract
We consider the class of crossed products of Noetherian domains with universal enveloping algebras of Lie algebras. For algebras from this class we give a sufficient condition for the existence of projective non-free modules. This class includes Weyl algebras and universal envelopings of Lie algebras, for which this question, known as the non-commutative Serre's problem, has been extensively studied previously. It turns out that the method of lifting of non-trivial stably free modules from simple Ore extensions can be applied to crossed products after an appropriate choice of filtration. The motivating examples of crossed products are provided by the class of relativistic internal time algebras, originating in non-equilibrium physics.

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1. Introduction

In [21], J-P Serre posed the question of whether any finitely generated projective module over the ring of commutative polynomials $\mathbf{k}[x_1, \ldots, x_n]$ over a field $\mathbf{k}$ is free. It was stated there, in geometrical language: whether any locally trivial vector bundle over an affine space $\mathbb{A}^n_{\mathbf{k}}$ is a trivial bundle. After almost 20 years of attempts, Suslin [24] and Quillen [20] independently (and using different methods) obtained an affirmative answer to Serre's question (see also [12] for a detailed study of the techniques involved).

Later on, this question was investigated for various classes of non-commutative rings, including those arising in physics, for example, relativistic internal time (RIT) algebras. A report on this work can also be found in Lam's book [12] (chapter VII.8). To describe briefly what has been done let us recall some definitions.

A finitely generated left A-module M is called *stably free* if $M \oplus A^n = A^m$ for some nonnegative integers n and m; clearly, it is then projective. A module, which is stably free but not free, will be called *non-trivial stably free*. We will need the *rank* of the stably free module, M, which is defined as $\mathrm{rk}M = m - n$. This definition obviously only makes sense if A is an invariant basis number (IBN) ring; for example, we may consider Noetherian rings. We will suppose throughout the paper that all rings are IBN.

The situation in the non-commutative case turned out to be more involved: there were constructed counterexamples, i.e. stably free non-free modules in several classes of non-commutative algebras. By saying that there exists a counterexample in a certain class of algebras, we mean that any algebra from this class allows a finitely generated projective but non-free module.

For example, stably free non-free ideals were constructed in any Weyl algebra, A_n, by Webber [25]. Another counterexample was constructed by Ojanguren and Sridharan [19] in rings of polynomials on two variables over a division ring (which is not a field). In group algebras, non-free projective modules were constructed by Dunwoody and Berridge [9] for torsion free groups and by Artamonov [7] for solvable groups. Examples of this type in enveloping algebras of non-Abelian finite-dimensional Lie algebras were provided by Artamonov and by Stafford; in [22], a unified way for producing non-trivial stably free right ideals was given, which virtually covers above cases.

In this paper, we consider the class of crossed products of Noetherian domains with a universal enveloping algebra of a Lie algebra, which subsumes most of the classes mentioned above, and provide a sufficient condition for the existence of stably free non-free modules in this wider class. More precisely, we show in theorem 6.2 that stably free non-free modules can be lifted from any subalgebra of the crossed product $A \star U\mathcal{G}$ which is a simple differential Ore extension $A[g,\delta]$, $g \in \mathcal{G}$, $\delta \in \mathrm{Der}A$ ($\delta = \delta_{\bar{g}}$ is a derivation, involved in the given crossed product, associated with the element $g \in \mathcal{G}$).

A useful tool is provided by theorem 5.7: if A is a domain, endowed with a filtration by a well-ordered semigroup, such that A_0 is a faithfully flat A-module, then any non-trivial stably free ideal in A_0 could be lifted to a non-trivial stably free ideal in A. A graded version of this fact (theorem 5.3) can be proved for a graded domain A graded by an *order-like semigroup*. This is a wider class of semigroups, which, however, captures most of the essential properties of well-ordered semigroups.

Let us emphasize that all examples of non-trivial stably free modules mentioned above, and just about all known examples in the non-commutative case, are modules of rank one. Over commutative rings, there are examples of higher rank and these are typical. An example of a module of minimal rank (over commutative rings) is a module of rank two over the ring, $\mathbb{R}[x,y,z]/x^2+y^2+z^2=1$. The way to ensure that the right unimodular row (x,y,z) gives rise to the non-trivial stably free module has a geometrical flavour (using the theorem of the 'hedgehog brushing' on a 2-sphere) and does not explain much in the line of techniques we study here.

For the class of Weyl algebras $A_n(k)$ it was proved by Stafford [23] that all stably free modules of rank two and bigger are free. In the proof, of course, the simplicity of $A_n(k)$ plays a crucial role. This result was generalized to some crossed products of simple rings with supersolvable groups by Jaikin-Zapirain in [11]. Another case where the positive result holds can be found in [6].

The examples of crossed products with the universal enveloping algebra we consider are provided by the class of RIT algebras, which we have been studying in [1, 2] and recall in section 8. This class originated in non-equilibrium physics [3, 5] and we consider it here in the general setting of crossed products.

2. Choice of subalgebras

We start with the definition of a crossed product with a universal enveloping algebra. By $\mathbf{k}$ we denote any field.

Definition 2.1. *Let A be a (non-commutative) $\mathbf{k}$-algebra, and $\mathcal{G}$ a Lie algebra with a basis $\{g_i | i \in I\}$ over $\mathbf{k}$. Then a $\mathbf{k}$-algebra B containing A is called a crossed product provided there is an embedding $\mathcal{G} \hookrightarrow B : g \mapsto \bar{g}$ of linear spaces, which satisfies:*

(1) $\bar{g}r - r\bar{g} \in A$ for any $g \in \mathcal{G}, r \in A$ and $\bar{g}r - r\bar{g} = \delta_g(r)$ is a derivation on A.
(2) $\bar{g}\bar{h} - \bar{h}\bar{g} = \overline{[g,h]}$ mod A for all $g, h \in \mathcal{G}$.
(3) B is a (left) free A-module with the commutative monomials $\bar{g}_1^{j_1} \ldots \bar{g}_m^{j_m}$ on $\{g_i | i \in I\}$ as a basis.

The latter condition (3) is called a PBW property over A.

We denote the crossed product algebra B by $A \star U\mathcal{G}$.

It is known from [22] (see also [13]) that non-trivial stably free ideals do exist in simple differential Ore extensions of Noetherian domains, which satisfy some additional condition.

In this section, we suggest how to choose appropriate subalgebras in a crossed product, $A \star U\mathcal{G}$, in such a way that their non-trivial stably free modules can be lifted to non-trivial stably free modules over the whole crossed product. Namely, we take subalgebras isomorphic to a simple Ore extension of the initial algebra A; thus the idea is to use subalgebras in the 'intersection' of the crossed product components.

We prove several properties of these subalgebras in order to prepare a tool which allows us to lift non-trivial stably free modules from these subalgebras to the whole crossed product.

Directly from the definitions, it can be seen that $A \star U\mathcal{G}$, where $\mathcal{G} = \{g\}$ is a one-dimensional Lie algebra, is isomorphic to the simple differential Ore extension $A[x, \delta]$, where $\delta = \delta_g$ is the derivation related to g, which was defined above as $\delta_g(r) = g\bar{r} - \bar{r}g$, for $r \in A$.

Using the defining relations (1) and (2) in the crossed product, one can easily see that $A \star U\mathcal{G}_1$, where $\mathcal{G}_1 = \{g\}$ is a Lie algebra generated by any single element $g \in \mathcal{G}$, is a subalgebra in $A \star U\mathcal{G}$. Indeed, the free basis of the A-module $A \star U\mathcal{G}_1$, according to (3) consists of elements $\bar{g}^i, i = 0 \leqslant i < \infty$. Any product of two elements of the shape $a\bar{g}^i, a \in A, g \in \mathcal{G}$ is a linear combination of elements of the same shape after applying (1).

We denote by A_1 the subalgebra $A \star U\mathcal{G}_1$ in $B = A \star U\mathcal{G}$ and will consider B as a left A_1-module writing ${}_{A_1}B$.

3. Faithful flatness of ${}_{A_1}B$

In this section we will prove two crucial properties of subalgebras of our choice which allow us to lift non-trivial stably free modules from $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$ to the whole crossed product $A \star U\mathcal{G}$.

Starting from here we suppose that A is a Noetherian domain.

The first property we need is the faithfully flatness of B as a left A_1 module. We will prove that in our situation even a stronger condition holds, namely

Lemma 3.1. *The left A_1-module ${}_{A_1}B$ is free.*

Proof. By definition of a crossed product, $B = A \star U\mathcal{G}$ and $A_1 = A \star U\mathcal{G}_1$ are free left A-modules with bases $V = \{g_1^{i_1} \ldots g_n^{i_n}\}$ and $W = \{g_1^i\}$, respectively.

We prove that ${}_{A_1}B$ is generated by the set $\Omega = \{g_2^{j_2} \ldots g_n^{j_n}\}$ and this set forms a free basis of this A_1-module.

The first part of the statement, saying that Ω is a generating system, is obvious. To show that this is a free basis it is enough to check that if $\Sigma a_i b_i = 0$ in B, for $a_i \in A_1, b_i \in \Omega$, then all $a_i = 0$ (since A_1 is a domain). To ensure this, let us first write elements $a_i \in A_1$ from the sum $\Sigma a_i b_i$ above as follows: $a_i = a_i(g_1) = \Sigma \alpha_k^{(i)} g_1^k$, with $\alpha_k^{(i)} \in A$.

Now fulfil the multiplication in the above sum and gather terms near each element v_i from V. We get $\Sigma \beta_j v_j = 0$, where $\beta_j = \Sigma_{i,k:v_j = g_1^k v_i} \alpha_k^{(i)}$. From this it follows that $\beta_i = 0$, since V is a free generating system for ${}_A B$. Since V is a free basis for the A-module B, given the fixed numbers i and j, there is only one k such that $v_j = g_1^k v_i$. Hence the set $\{\beta_j\}$ just coincides with the set $\{\alpha_k^{(i)}\}$. So, together with all $\beta_i = 0$ we have all $\alpha_k^{(i)} = 0$, and hence $a_i = 0$ for all i. □

4. Strongly completely prime subalgebras

Before we start the discussion of the second main lemma we should introduce the notion of *strongly completely prime subalgebra*, or *s.c.p.-subalgebra* for short.

Let us consider the following two properties of subalgebra.

Definition 4.1. *We say that a subalgebra A_1 is completely prime in A if for any two non-zero elements a and b from A, $ab \in A_1$ implies $a \in A_1$ or $b \in A_1$.*

Definition 4.2. *We say that a subalgebra A_1 is strongly completely prime (s.c.p.) in A if for any two non-zero elements a and b from A, $ab \in A_1$ implies $a \in A_1$ and $b \in A_1$.*

In case A_1 is an ideal in A, the first definition just coincides with the definition of a completely prime ideal, i.e. an ideal such that the quotient is a domain (this explains our terminology).

The second definition degenerates in case A_1 is an ideal. Indeed, suppose $A_1 \triangleleft A$, $A_1 \neq \{0\}$ and $A \backslash A_1 \neq \emptyset$. Then take an element $b \in A \backslash A_1$, since A_1 is a (right) ideal, for an arbitrary non-zero element $a \in A_1$ we have $ab \in A_1$, and the property from the definition 2 does not hold. If $A_1 = \emptyset$, then formally the property of being $s.c.p.$ always holds in a domain.

So, the property of being $s.c.p.$ is clearly a feature of subalgebras and should be considered only in this case (rather then for ideals).

Let us discuss now the notion of type of an element in the crossed product.

First, we associate with any product (monomial) $w = a g_{i_1} \dots g_{i_n}, i_k \in I, a \in A$ in the crossed product $B = A * \mathcal{U}(\mathcal{G})$ its type on variables $g_{i_1} \dots g_{i_n}$. By definition, *the type $t(w)$* of the element w is a tuple of nonnegative integers $(j_1, \dots, j_r)$, where j_k is the number of variables g_k in the monomial w for any $k \in I$. (In case $j_l = 0$ for all $l > r$, we just omit zero terms in the sequence $j_1, \dots, j_r, \dots$ starting from j_{r+1} to get $t(w)$). One can also consider the type of a monomial on any subset of variables $\{g_{i_k}, i_k \in I' \subset I\}$.

Let us fix an order on monomials $w = a g_{i_1} \dots g_{i_n} \in B$ using the degree lexicographical ordering on commutative words $t(w)$. Namely, we say that $w > w'$ for $w = a g_{i_1} \dots g_{i_n}$ and $w' = a' g_{i'_1} \dots g_{i'_n}$ if $t(w) >_{dl} t(w')$. The latter means that if $t(w) = (j_1, \dots, j_r)$ and $t(w') = (j'_1, \dots, j'_s)$, then either $r > s$ or $r = s$ and $j_t > j'_t$ for some t, such that $j_l = j'_l$ for all $l < t$.

We can define a *normal form* (with respect to $g_i, i \in I$) of an element in $B = A * \mathcal{U}(\mathcal{G})$. We say that an element $f = \sum a_{\bar{i}} g_{i_1} \dots g_{i_n}$ is *in the normal form* if $i_1 \leqslant i_2 \leqslant \dots \leqslant i_n$, $a_{\bar{i}} \in A$, that is, all monomials have the form $g_1^{j_1} \dots g_r^{j_r}$. It is clear from the relations in the definition of crossed product that any element from B can be presented in a normal form, since these relations allow the commutation of $r \in A$ with $g \in \mathcal{G}$ and elements from $\mathcal{G}$ between each other. In both cases, we might get new terms, which have a lower degree in $g_i, i \in I$. Since there is no infinite chain of words in g_i of strictly increasing degree, in certain step we will get an element equal to w in a normal form. This element we will call a *normal form of* $w \in B = A * \mathcal{U}(\mathcal{G})$ and denote it by $\mathcal{N}(w)$. Property (3) in the definition of the crossed product (PBW property) ensures that the normal form of an element in B is unique.

This allows us to introduce the notion of the type of an element $b \in B$.

Definition 4.3. *By the type of an arbitrary element $b \in B$, we call the type of the highest monomial in the normal form of b.*

Having in hands the notion of the type of an element in $B = A \star U\mathcal{G}$ we actually have a natural filtration on B. Namely, $B = \cup_{\bar{i}\in\Sigma_n} B_{\bar{i}}$, where Σ_n is a semigroup of tuples $(i_1, \ldots, i_n)$ with the componentwise operation, and $B_{\bar{i}}$ is a linear span over $\mathbf{k}$ of elements of the type $(i_1, \ldots, i_n)$, in particular, $B_0 = A$.

The existence of such a filtration forces us to develop a general machinery for the graded and filtered case and then apply it to the situation of crossed products, using however a filtration different from that above.

5. Semigroup graded and filtered case

For this section, we break our agreement that A is a domain, in some statements here we will ask only for A being a graded domain (that is, there are no zero divisors among homogeneous elements with respect to a given grading).

The main theorem in the graded setting will have the form.

Theorem 5.1. *Let $A = \bigoplus_{j\in\mathbb{Z}_+} A_j$ be a $\mathbb{Z}_+$-graded domain, where A is a flat A_0-module. Then any stably free non-free module over A_0 can be lifted to A.*

This theorem can be further generalized in a sense that one can consider gradings more general than $\mathbb{Z}_+$-gradings. We shall prove a theorem in that bigger generality, so theorem 5.1 will follow from theorem 5.3.

Definition 5.2. *We call a semigroup $(G, +)$ ordered-like if it has no invertible elements except 0 and for any two finite subsets S_1, S_2 of G such that $S = S_1 + S_2 \neq \{0\}$, there exists $c \in S$ with*

$$\nu(c) = |\{(a, b) : a \in S_1, b \in S_2, a + b = c\}| = 1.$$

Most common examples of semigroups with such a property are well-ordered semigroups, allowing for a linear order compatible with an operation: $a < b \Longrightarrow a + c < b + c$. In this case, the sum of maximal elements of S_1 and S_2 will serve as an element, $c \in S$, with unique presentation. But there are other examples where this property does not come from well-ordering.

Theorem 5.3. *Let $A = \bigoplus_{\sigma\in G} A_\sigma$ be a domain graded by an ordered-like semigroup G, where A is faithfully flat as a left A_0-module. Then any stably free non-free right ideal in A_0 can be lifted to A.*

The proof is based on the following lemmas.

Lemma 5.4. *Let $A = \bigoplus_{\sigma\in G} A_\sigma$ be a graded domain, with G being an ordered-like semigroup. Then A_0 is a completely prime subalgebra of A.*

Proof. To prove that A_0 is completely prime it is enough to ensure that, for any two elements $a, b \in A$, from $a, b \notin A_0$ it follows that $ab \notin A_0$. For any element a of A let us denote by $S(a)$ the subset $S(a) = \{\sigma \in G : a_\sigma \neq 0\}$ of the semigroup G, where $a = \sum_{\sigma \in G} a_\sigma, a_\sigma \in A_\sigma$ is the graded decomposition of a. Clearly $a = \sum_{\sigma \in S(a)} a_\sigma$. Since $a, b \notin A_0$, the sets $S(a)$ and $S(b)$ contain non-zero elements. Since G has no non-zero invertible elements, we have $S = S(a) + S(b) \neq \{0\}$. Taking into account that G is ordered like, we can find $\gamma \in S \setminus \{0\}$ such that $\nu(\gamma) = |\{(\sigma, \tau) : \sigma \in S(a), \tau \in S(b), \sigma + \tau = \gamma\}| = 1$. For the γth graded component of ab we will have $(ab)_\gamma = a_\sigma b_\tau$. Since $a_\sigma \neq 0, b_\tau \neq 0$ and A is a graded domain, we get $(ab)_\gamma \neq 0$ for $\gamma \neq 0$. So, $ab \notin A_0$. □

The following fact is true for grading by any semigroup, not necessarily with the ordered-like property.

Lemma 5.5. *Let $A = \bigoplus_{\sigma \in G} A_\sigma$ be a domain graded by an arbitrary Abelian semigroup G. Then the property of A_0 to be completely prime implies the property of A_0 to be strongly completely prime.*

Proof. To ensure this we should show that if $a, b \in A \setminus \{0\}$ and $ab \in A_0$ implies $a \in A_0$, then we also have $b \in A_0$.

Indeed, let $b = \sum_{g \in S(b)} b_g$ be the graded decomposition of b. Then $ab = \sum_{g \in S(b)} ab_g$. Here $(ab)_g = ab_g \in A_0 A_g \subseteq A_g$. On the other hand, $ab \in A_0$ and therefore $(ab)_g = ab_g = 0$ for any $g \neq 0$. Since $a \neq 0$ and A is a graded domain, this implies that $b_g = 0$ for any $g \neq 0$. That is, $b \in A_0$. □

As a corollary of lemmas 5.4 and 5.5 we have that the subalgebra A_0 of A is strongly completely prime. Using this we can proceed with the proof of theorem 5.3 by analogy with [22].

Proof. (of theorem 5.3) Let K be a non-trivial stably free right ideal in A_0. We will show that the induced ideal, $KA = K \otimes_{A_0} A$, is also stably free but not free.

Since A is flat as the A_0-module, $KA = K \otimes_{A_0} A$ and is also projective as the A-module, hence stably free. The essential part is to prove that it is not free. We have $KA \oplus A = A \oplus A$, thus we have to show that KA is not cyclic.

Suppose this is not the case, i.e. $KA = yA$ for some $0 \neq y \in A$. (In case $y = 0$ we will have a contradiction immediately: $A = yA \oplus A = A \oplus A$ and this contradicts the condition we suppose to hold throughout the paper that all rings have the IBN property).

Since $KA = K \otimes_{A_0} A$, K is the A_0 submodule in KA: $K \subset KA = yA$, and we can take a nonzero element $p \in K$, which is $p = yb$ for some nonzero $b \in A$. But K is an ideal in A_0 and we can use primeness of A_0: if $p \in A_0$ and $p = yb$ for nonzero $y, b \in A$ then it should imply $y \in A_0$.

Now, for any right ideals $I \subsetneqq J \triangleleft_r A_0$, due to faithfully flatness of A we have $IK \subsetneqq JK \triangleleft_r A$.

Suppose that the following inclusion of right ideals in A_0 holds: $K \subsetneqq KA \cap A_0$. Then applying the above observation we get $KA \subsetneqq (KA \cap A_0)A$. But in fact $(KA \cap A_0)A = KA$. Indeed, $K \subset KA \cap A_0$ implies $KA \subset (KA \cap A_0)A$, while $KAA \cap A_0 A \subset KA \cap A = KA$. This contradiction shows that $K = KA \cap A_0$.

Now we have $K = KA \cap A_0 = yA \cap A_0 \subset yA_0$, due to $yb \in yA$ belongs to A_0 we again use the fact that A_0 is a prime subalgebra, this implies $b \in A_0$, in case $b \neq 0$ (obviously in case $b = 0$ we also have $b \in A_0$). Hence $K = yA \cap A_0 = yA_0$; this contradiction completes the proof. □

Remark. Let us mention here that for the question of existence of non-trivial stably free modules it is enough to look only at ideals. In other words, the existence of a non-trivial stably free (right) module is equivalent to the existence of a non-trivial stably free (right) ideal. This follows from the observation that if we have a f.g. projective module, which is non-free, then we also have a projective non-free ideal. Indeed, let P be a f.g. projective R-module and $R^n = P \oplus Q$. For any $x \in R^n$ there is a unique decomposition, $x = x_P + x_Q$. Consider a submodule of R^n of the form $(0) \times \cdots \times (0) \times R \times (0) \times \cdots \times (0) = R_j \subset R^n$, and define with respect to the above decomposition, submodules $P_j = \{x_P | x \in R_j\} \subset P$ and $Q_j = \{x_Q | x \in Q_j\} \subset Q$. Clearly, we have an isomorphism $R_j \simeq P_j \oplus Q_j$. On the other hand, from the definition of R_j it is clear that $R_j = R$. Moreover, we have $P = P_1 \oplus \cdots \oplus P_n$ and if P is not free, then one of P_j is not free. But P_j is a submodule of R, i.e. an ideal in R. So we get a projective ideal which is non-free. This remark shows why it is enough to consider in the theorem only the behaviour of ideals under the extension of the base ring.

Now we will formulate filtered versions which will be used for the results about crossed products. Here we restrict ourselves to an arbitrary well-ordered semigroup.

Lemma 5.6. *Let $A = \cup_{\sigma \in \Sigma} U_\sigma$ be a domain, endowed with a filtration by a well-ordered semigroup Σ. Then U_0 is a s.c.p.-subalgebra of A.*

Theorem 5.7. *Let $A = \cup_{\sigma \in \Sigma} U_\sigma$ be a domain, endowed with a filtration by a well-ordered semigroup Σ, and A is faithfully flat as a left U_0-module. Then any stably free non-free right ideal in U_0 can be lifted to A.*

Proofs are analogous to those of lemmas 5.4, 5.5 and theorem 5.3.

6. Back to crossed products

Now we can prove the second lemma we need in the cross product case.

Lemma 6.1. *A subalgebra $A_1 = A \star U\mathcal{G}_1$ in $B = A \star U\mathcal{G}$, where $\mathcal{G}_1 \subset \mathcal{G}$ is a Lie subalgebra of $\mathcal{G}$ generated by one (nonzero) element, is a s.c.p.-subalgebra.*

Proof. The essential point in this proof is an appropriate choice of filtration on B. After that we apply lemma 5.6. Instead of using a natural filtration on B mentioned at the end of section 4, we suggest the following one. Let

$$B = \bigcup_{\bar{i} \in \Sigma_{n-1}} B_{\bar{i}},$$

where

$$B_{\bar{i}} = A[g_1] U_{\bar{i}},$$

for

$$U_{\bar{i}} = \mathrm{Sp}\langle g_2^{i_2} \ldots g_n^{i_n} \big| \bar{i} = (i_2, \ldots, i_n) \in \Sigma_{n-1} \rangle_{\mathbf{k}},$$

in particular, $B_0 = A[g_1]$ is a polynomial algebra over A on one variable g_1, where $\mathcal{G}_1$ generated by g_1 (as we set in section 2).

Note that it is a filtration by the well-ordered semigroup Σ_{n-1} and an order on it is degree lexicographical (the same as we used for ordering of types in section 4, but this time with respect to $n - 1$ variables $g_2, \ldots, g_n$). It is an easy exercise then to check that this is indeed a filtration. □

Using tools provided by the lemmas 3.1 and 6.1 we can lift non-trivial stably free modules from subalgebras of the type $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$ to the crossed product.

Theorem 6.2. *Let $B = A \star U\mathcal{G}$. Let K be a non-trivial stably free right ideal in $A_1 = A \star U\mathcal{G}_1 = A[g, \delta]$, for some $g \in \mathcal{G}$. Then the induced right ideal, $K \otimes_{A_1} B$, in B is stably free, but not free.*

Proof. We use the same filtration as in the previous lemma and apply theorem 5.7 together with lemmas 3.1 and 6.1. □

The lifting technique could be applied whenever we have a $s.c.p.$-subalgebra D in B, such that ${}_DB$ is a faithfully flat module. Lemmas 3.1 and 6.1 ensure that it is always the case for the crossed product algebra $B = A \star U\mathcal{G}$, if we choose as a subalgebra D a simple Ore extension $A[g, \delta]$ of A.

Now we are in a position to state the result which gives a sufficient condition of existence of non-trivial stably free modules over crossed products.

Theorem 6.3. *Let A be a Noetherian domain, $U\mathcal{G}$—the universal enveloping of Lie algebra $\mathcal{G}$ and $B = A \star U\mathcal{G}$ a crossed product. If there exists an element $g \in \mathcal{G}$ such that $(r, g + q)$ is a unimodular row in a subalgebra $A[g, \delta]$ of B, for some $r, q \in A$, r a non-unit, then the ideal $rB \cap (g + q)B$ is a non-trivial stably free B-module.*

This result shows that non-trivial stably free modules can be lifted from the Ore extensions of the basic ring A, appearing inside the construction of the crossed product with the universal enveloping algebra.

Obviously, these modules do not always exist over $A \star U\mathcal{G}$. This we can already see from the example of a simple Ore extension $A[g, \delta]$, which is also the simplest case of a crossed product. Take A to be a commutative local ring with the maximal ideal $\mathcal{M}$. It is known (see [22]) that a non-trivial Ore extension of A allows stably free non-free ideals if and only if at least one of the following conditions fails: (1) the Krull dimension of the basic ring is one: $\mathrm{Kdim} A = 1$ or (2). $\delta(\mathcal{M}) \subseteq \mathcal{M}$. Thus the situation in the wider class of crossed products is not so definitive as in group algebras of solvable groups or in Weyl algebras where non-trivial stably free modules always exist, so we only can give conditions when they do.

As an immediate consequence of the above-mentioned fact and theorem 6.2 we get.

Corollary 6.4. *A crossed product of a local commutative ring A of $\mathrm{Kdim} A > 1$ with $U\mathcal{G}$ for an arbitrary Lie algebra G always allows a stably free non-free module. If $\mathrm{Kdim} A \leqslant 1$, then the non-trivial stably free module does exist if $\mathcal{G}$ acts in such a way that for some $g \in \mathcal{G}$, $g(\mathcal{M}) \not\subset \mathcal{M}$, where $\mathcal{M}$ is a maximal ideal in A.*

7. Remark on modules of higher ranks

Here we recall some known results, just to emphasize that in the class of crossed products there are obviously examples of non-trivial stably free modules of higher ranks. They can be obtained by a slight modification of arguments for the case of 2-sphere (see [13], 11.2.3).

Namely, let us take a (commutative) ring $A = \mathbb{R}[x_1, \ldots, x_n]/\sum_{i=1}^n x_i^2 - 1$, for $n \geqslant 3$. Due to the nature of these relations the column $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, with entries a_i—images of variables x_i under the natural morphism $\varphi : \mathbb{R}[x_1, \ldots, x_n] \to A$, is unimodular, that is $Aa_1 + \cdots + Aa_n = A$.

Hence it defines a split monomorphism

$$\alpha : A \hookrightarrow A^n : a \mapsto \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot a,$$

with cokernel P, so $P \oplus A = A^n$. Suppose that P is a free A-module. This is equivalent to the fact that the column $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ is extendable to an invertible matrix. That is, there exists $M \in Gl_n(\mathbb{R}), M = (\bar{r}, \bar{c}_1, \ldots, \bar{c}_{n-1})$, where $\bar{r}, \bar{c}_1, \ldots, \bar{c}_{n-1}$ denote columns of the matrix and $\bar{r} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. We can construct a continuous tangent vector field on a sphere $\mathbb{S}^{n-1}$, it is provided by the minors $v_r = M_{i2}(r)$ of matrix M, corresponding to the second column. Indeed, the scalar product $(r, v_r) = \det(\bar{r}, \bar{r}, \bar{c}_2, \ldots, \bar{c}_{n-1}) = 0$. On the other hand, this vector field cannot vanish, since there exists a vector $\bar{c}_1$, such that $(c_1, v_r) = \det(r_1, c_1, c_2, \ldots, c_{n-1}) = \det(\mathrm{M}) \neq 0$.

The existence of a continuous tangent vector field on a real $n - 1$ sphere which vanishes nowhere does contradict, for even n, with the well-known theorem on the 'brushing of a hedgehog' (or 'hairy ball theorem'; see, for example, [14]).

8. The relativistic internal time algebra

In this section, we will show how our general results work for the example of the RIT algebra. The former was a motivating example to clarify the general construction of cross products.

In the series of papers [10, 17, 18], an *internal time operator* for a unitary evolution group $U_t, t \in \mathbb{R}$ on a separable Hilbert space $\mathcal{H}$ was introduced and studied. By definition it is a self-adjoint operator T with domain $\mathcal{D}$ on which the following property holds: $U_{-t} T U_t = T + tI$. For the unitary group $U_t = \mathrm{e}^{P_0 t}$, where P_0 is an anti self-adjoint operator, this definition boils down to the commutation relation $[P_0, T] = -I$. An internal time operator was introduced by Misra [15] in the context of unstable Kolmogorov dynamical systems.

The study of the time operator for *relativistic fields* [3–5, 16] leads naturally to consideration of the RIT Lie algebra $\mathcal{L}$ which is an infinite-dimensional modification of the Poincaré algebra, generated by the *time operator* T and the ten generators of the Poincaré group, describing a relativistic system: P_0 is the time evolution generator, P_α are the momenta generating space translations, J_α are the angular momenta generating rotations in space, and N_α are the Lorentz boost generators, which are spatio-temporal rotations in Minkowski space.

The internal time gives rise to the velocity observable $V_\alpha = P_\alpha P_0^{-1}$ and to the internal position observable $T V_\alpha$. Thus the algebra of relativistic system with internal time does not satisfy the Einstein equations any more and associated internal spacetime is not the Minkowski spacetime. Concrete commutation relations for the substitute $\mathcal{L}$ of the Poincaré Lie algebra, with internal time, were computed [3, 5].

We consider [1, 2] associated graded algebra of an enveloping of the Lie algebra $\mathcal{L}$ and say that the *associative* RIT *algebra of type* (10, 4) is an algebra given by the following commutation relations:

$$R = \mathbf{k}\langle E\rangle / \begin{cases} [e_i, e_j] = 0, & \forall (e_i, e_j) \in (E \times E) \setminus \{(N_\alpha, V_\beta), (N_\alpha, T)\}, \\ [N_\alpha, T] = V_\alpha T, \\ [N_\alpha, V_\beta] = -V_\alpha V_\beta, \end{cases}$$

where $E = \{P_0, P_\alpha, J_\alpha, N_\alpha, T, V_\alpha | \alpha = 1, 2, 3\}$.

For this particular algebra, the question of existence of stably free non-free modules was considered in [1]. Using the general result of this paper, the following series of examples of stably free non-free modules over R could be obtained. Let E_1 be the set of variables $E_1 = \{E \setminus N_\alpha\}$, for a fixed $\alpha \in \{1, 2, 3\}$, and S is a differential Ore extension of $\mathbf{k}[E_1]$: $S = \mathbf{k}[E_1][N_\alpha, \delta_\alpha]$, where the derivation δ_α is defined by the commutation relations. Then applying the technique developed above we can obtain that the induced ideal $K \otimes_S R$ for $K = (V_\alpha + 1)S \cap N_\alpha S$ is a stably free non-free right ideal in R.

Remark on non-gradable modules. As it is noted in [8], the class of associative RIT algebras consists of Auslander regular algebras. From this it follows that non-trivial stably free modules, we construct here, are also examples of non-gradable modules.

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